

Damaskinsky E.V.¹, Sokolov M.A.²

The generating function of bivariate Chebyshev polynomials associated with the Lie algebra G_2 ³

Dedicated to the memory of our dear friend Peter Kulish

The generating function of the second kind bivariate Chebyshev polynomials associated with the simple Lie algebra G_2 is constructed by the method proposed in [1] and [2].

1 Introduction

The present work finishes the series of the articles ([1], [2], [3]) which were initiated by P. Kulish. In these works we propose the method of constructing of generating functions for Chebyshev polynomials in several variables (of the first and second kinds), associated with simple Lie algebras. The method was tested on examples of polynomials related to Lie algebras A_2 , C_2 , A_3 and G_2 (in the last case only for the polynomials of the first kind). In this work we consider the case of bivariate Chebyshev polynomials of the second kind associated with the algebra G_2 , thus complete the consideration of polynomials related with simple Lie algebras of rank 2.

Attention of P. Kulish in this subject arose during the preparation of his report at the conference in Nankai [4]. During the discussion of the report with one of the authors (EVD), it was found that the Chebyshev polynomials of the 2-nd kind of one or two variables naturally arise when considering the integrable spin chains by the quantum inverse scattering method.

It was assumed that this topic will be discussed in detail in further studies, however, the illness and death of P. Kulish prevented the implementation of this plan.

In this paper we construct generating function of bivariate Chebyshev polynomials of the second kind associated with the simple Lie algebra G_2 and make a few comments regarding this construction. We will consider the most natural variant of generalization of Chebyshev polynomials to the case of several variables suggested in the work of Koornwinder [5] (see also [6], [7]).

2 Description of the method

Chebyshev polynomials in several variables are a natural generalization of the classical Chebyshev polynomials of one variable (see for example [8], [9]). They have application in various areas of mathematics (in the theory of approximations [10], in the linear algebra [11], [12], in the theory of representations [13]-[15]) and in physics ([16]-[19]).

¹Math. Dept. Military Engineering Institute. VI(IT). Saint Petersburg, evd@pdmi.ras.ru

²St. Petersburg state Peter The Great Polytechnical University; masokolov@gmail.com

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The classical Chebyshev polynomials of the first kind $T_n(x)$ are defined by the formula

$$T_n(x) = T_n(\cos \phi) = \cos n\phi, \quad n \geq 0, \quad (1)$$

where $\phi = \arccos x$. Polynomials of the second kind $U_n(x)$ is defined as

$$U_n(x) = \frac{\sin(n+1)\phi}{\sin \phi}, \quad n \geq 0. \quad (2)$$

Both polynomials satisfy the well known three-term recurrence relation

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \quad n \geq 1, \quad (3)$$

but with different initial conditions

$$T_0(x) = 1, \quad T_1(x) = x, \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

Chebyshev polynomials of several variables can be determined for any simple Lie algebra in the following way. Let L be the simple Lie algebra with reduced root system R consisting of vectors in d -dimensional Euclidean space E^d with the scalar product (\cdot, \cdot) .

The root system R is completely determined by the basis of simple roots $\alpha_i, i = 1, \dots, d$ and finite reflection group $W(R)$ — the Weyl group. Generating elements of the Weyl group $w_i, i = 1, \dots, d$ acts on the simple roots by the rule $w_i \alpha_i = -\alpha_i$. The root system R is closed under the action of the Weyl group. For any vector $x \in E^d$ elements w of the group $W(R)$ acting according to the formula

$$w_i x = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i. \quad (4)$$

To each root $\alpha \in R$ correspond associated co-root

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

For the basis of simple co-roots $\alpha_i^\vee, i = 1, \dots, d$ one can define the dual basis of the fundamental weights $\lambda_i, i = 1, \dots, d$

$$(\lambda_i, \alpha_j^\vee) = \delta_{ij}$$

(the dual space E^{d*} is identified with E^d). The bases of the roots and weights are related by the linear transformation

$$\alpha_i = C_{ij} \lambda_j, \quad C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad (5)$$

where C is the Cartan matrix of the algebra L . Recall that the bases of roots and weights are not orthonormal.

The Chebyshev polynomials of the first kind associated with the Lie algebra L is defined via W -invariant functions

$$\Phi_{\mathbf{n}}(\phi) = \sum_{w \in W} e^{2\pi i(w\mathbf{n}, \phi)}, \quad (6)$$

where vector \mathbf{n} with nonnegative integer components expressed in the basis of the fundamental weights $\{\lambda_i\}$ and ϕ – in the dual basis of co-roots $\{\alpha_i^\vee\}$

$$\mathbf{n} \equiv (n_1, \dots, n_d) = \sum_{i=1}^d n_i \lambda_i \quad n_i \in N, \quad \phi = \sum_{i=1}^d \phi_i \alpha_i^\vee \quad \phi_i \in [0, 1).$$

Defining new variables (generalized cosines) x_i , $i = 1, \dots, d$ by relations

$$x_i = \Phi_{\mathbf{e}_i}(\phi), \quad \mathbf{e}_i = (\overbrace{0, \dots, 0}^{i-1}, 1, \overbrace{0, \dots, 0}^{d-i}). \quad (7)$$

and recording function (6) in terms of x_i we come to the (non-normalized) Chebyshev polynomials of the first kind in d variables. The fact that this procedure is possible for any simple Lie algebra, shown in the works [5], [20] - [24].

The function $\Phi_{\mathbf{n}}(\phi)$ has many useful properties, including the following "rule of multiplication"

$$\Phi_{\mathbf{k}} \Phi_{\mathbf{s}} = \sum_{w \in W} \Phi_{w\mathbf{k}+\mathbf{s}}, \quad \mathbf{k} = (k_1, \dots, k_d), \quad \mathbf{s} = (s_1, \dots, s_d), \quad (8)$$

which allows to find the recurrence relation for polynomials in several variables.

The Chebyshev polynomials of the second kind associated with the Lie algebra L , are determined by the relations

$$U_{\mathbf{n}}(\phi) = \frac{\sum_{w \in W} \det w \, e^{2\pi i(w(\mathbf{n}+\boldsymbol{\rho}), \phi)}}{\sum_{w \in W} \det w \, e^{2\pi i(w\boldsymbol{\rho}, \phi)}}, \quad (9)$$

where $\det w = (-1)^{\ell(w)}$ and $\ell(w)$ is the minimum number of generators w_i of the Weyl group w_i needed to represent the element of w by product of generators from w_i . In (9) $\boldsymbol{\rho}$ is the Weyl vector equal to the sum of positive roots. With the help of relations (5) this vector can be described in terms of the fundamental weights. As in the case of polynomials of the first kind, one can define new variables $x_i = U_{\mathbf{e}_i}(\phi)$, in terms of which can be described function (9) for any non-negative n_i from multi-index $\mathbf{n} = (n_1, \dots, n_d)$. The function $U_{\mathbf{n}}(\phi)$, presented in terms of x_i , gives the second kind Chebyshev polynomials in several variables.

In practice describing of the functions $\Phi_{\mathbf{n}}(\phi)$ (6) and $U_{\mathbf{n}}(\phi)$ (9) (with given multi-index $\mathbf{n} = (n_1, \dots, n_d)$) in terms of the variables x_i is quite cumbersome task. So obviously, one need an effective method of direct computation of Chebyshev polynomials in several variables. In the works [1], [2] we propose a new method for calculation of generating functions for Chebyshev polynomials of any kind associated with any simple Lie algebra. Let us describe it briefly.

Let's start with the function (6) that defines the Chebyshev polynomials of the first kind. Because components of the vector \mathbf{n} are all non-negative numbers, the scalar product in the functions $\Phi_{\mathbf{n}}(\phi)$ can be represented in the form $(w\mathbf{n}, \phi) = \sum_k w(\lambda_k, \phi) n_k$, and the function $\Phi_{\mathbf{n}}(\phi)$ can be expressed as the following products

$$\Phi_{\mathbf{n}} = \sum_{w \in W} \prod_k (e^{2\pi i(w\lambda_k, \phi)})^{n_k} = \text{tr} \left(\prod_k M_k^{n_k} \right). \quad (10)$$

Here M_k — diagonal matrices, nonzero elements of which are the components of the sum (6)

$$M_k = \text{diag}(e^{2\pi i(w_1 \lambda_k, \phi)}, e^{2\pi i(w_2 \lambda_k, \phi)}, \dots, e^{2\pi i(w_{|W|} \lambda_k, \phi)}), \quad (11)$$

where w_i — elements of the Weyl group W , and $|W|$ is the number of elements of this group.

We introduce a new matrix $R_k = (I_{|W|} - p_k M_k)^{-1}$, where $I_{|W|}$ is $|W|$ -dimensional identity matrix and p_k are arbitrary real parameters. In this notation the function $\Phi_{\mathbf{n}}$ has the form

$$\Phi_{\mathbf{n}}(\phi) = \Phi_{n_1, \dots, n_d}(\phi) = \frac{1}{n_1! \dots n_d!} \frac{\partial^{n_1 + \dots + n_d}}{\partial^{n_1} p_1 \dots \partial^{n_d} p_d} (\text{tr}(R_{p_1} \dots R_{p_d})) \Big|_{p_1 = \dots = p_d = 0}. \quad (12)$$

The simple structure of matrices R_{p_k} allows to represent the function $\text{tr}(R_{p_1} \dots R_{p_d})$ in terms of the variables defined above (7)

$$x_i = \Phi_{\mathbf{e}_i}(\phi), \quad \mathbf{e}_i = (\overbrace{0, \dots, 0}^{i-1}, 1, \overbrace{0, \dots, 0}^{d-i}).$$

The coefficients $\Phi_{n_1, \dots, n_d}(\phi)$ of the function

$$F_{p_1, \dots, p_d}^I = \text{tr}(R_{p_1} \dots R_{p_d}) = \sum_{n_1 \dots n_d \geq 0} \Phi_{n_1, \dots, n_d}(x_i) p_1^{n_1} \dots p_d^{n_d} \quad (13)$$

also can be expressed in terms of x_i and are therefore the (non-normalized) Chebyshev polynomials of the first kind in d variables. The function F_{p_1, \dots, p_d}^I is the generating function of these polynomials.

The computation of the generating functions for the second kind Chebyshev polynomials in d variables required only a small modification of the method described above. Consider a function $U_{\mathbf{n}}(\phi)$ that determines these polynomials in the form

$$U_{\mathbf{n}}(\phi) = \frac{\Phi_{\mathbf{n}+\boldsymbol{\rho}}^{as}}{\Phi_{\boldsymbol{\rho}}^{as}} = \frac{\Phi_{\mathbf{n}+\boldsymbol{\rho}}^{as+} - \Phi_{\mathbf{n}+\boldsymbol{\rho}}^{as-}}{\Phi_{\boldsymbol{\rho}}^{as+} - \Phi_{\boldsymbol{\rho}}^{as-}}, \quad (14)$$

where

$$\Phi_{\mathbf{k}}^{as\pm} = \sum_{w \in W, \det w = \pm 1} e^{2\pi i(w \mathbf{k}, \phi)}.$$

Further calculations, repeat the considerations given above (10) - (13) for each function $\Phi_{\mathbf{k}}^{as\pm}$ separately. In the result the generating function for the polynomials of the second kind takes the form

$$F_{p_1, \dots, p_d}^{II} = \frac{\text{tr}(R_{p_1}^+ \dots R_{p_d}^+ - R_{p_1}^- \dots R_{p_d}^-)}{\Phi_{\boldsymbol{\rho}}^{as}}, \quad (15)$$

where the matrix R^{\pm} are expressed in terms of the functions $\Phi_{\mathbf{k}}^{as\pm}$. Since the Weyl vector in the space of the fundamental weights is given by the formula $\boldsymbol{\rho} = \sum_{i=1}^d \lambda_i$, the multiplier $\Phi_{\boldsymbol{\rho}}^{as}$ at the denominator of (14), is calculated as

$$\Phi_{\boldsymbol{\rho}}^{as} = \frac{\partial^d}{\partial p_1 \dots \partial p_d} (\text{tr}(R_{p_1}^+ \dots R_{p_d}^+ - R_{p_1}^- \dots R_{p_d}^-)) \Big|_{p_1 = \dots = p_d = 0}. \quad (16)$$

Thus the Chebyshev polynomials of the second kind are given by the expressions

$$U_{\mathbf{n}} = \frac{1}{(n_1 + 1)! \dots (n_d + 1)!} \frac{\partial^{n_1 + \dots + n_d + d}}{\partial^{n_1+1} p_1 \dots \partial^{n_d+1} p_d} \left(\frac{\text{tr}(R_{p_1}^+ \dots R_{p_d}^+ - R_{p_1}^- \dots R_{p_d}^-)}{\Phi_{\boldsymbol{\rho}}^{as}} \right) \Big|_{p_1 = \dots = p_d = 0}. \quad (17)$$

3 The Chebyshev polynomials of the second kind associated with the Lie algebra G_2

Using the above scheme, we calculate the generating function for bivariate Chebyshev polynomials of the second kind, associated with the exceptional Lie algebra G_2 . The root system of this algebra has two fundamental roots α_1, α_2 and includes the following positive roots $\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$ along with their reflections.

Using the Cartan matrix

$$C_{G_2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

and the formula (5), we obtain the action of the generating elements w_1, w_2 of the Weyl group $W(G_2)$ on the fundamental roots and fundamental weights

$$w_1\alpha_1 = -\alpha_1, \quad w_1\alpha_2 = 3\alpha_1 + \alpha_2, \quad w_2\alpha_1 = \alpha_1 + \alpha_2, \quad w_2\alpha_2 = -\alpha_2.$$

$$w_1\lambda_1 = \lambda_2 - \lambda_1, \quad w_1\lambda_2 = \lambda_2, \quad w_2\lambda_1 = \lambda_1, \quad w_2\lambda_2 = 2\lambda_1 - \lambda_2.$$

The expression of other elements of Weyl groups via generators looks as

$$w_3 = w_1w_2, \quad w_4 = w_2w_1, \quad w_5 = w_2w_1w_2, \quad w_6 = w_1w_2w_1, \quad w_7 = (w_1w_2)^2,$$

$$w_8 = (w_2w_1)^2, \quad w_9 = w_2(w_1w_2)^2, \quad w_{10} = w_1(w_2w_1)^2, \quad w_{11} = (w_1w_2)^3, \quad w_0 = e,$$

and allows to extend the above relations to the whole group. Determinants of the elements $w_1, w_2, w_5, w_6, w_9, w_{10}$ of the Weyl group is equal to -1 , the determinant of the remaining elements are equal to unit.

Using the above relations and the definition

$$\Phi_{\mathbf{k}}^{as} = \sum_{w \in W} \det w e^{2\pi i(w\mathbf{k}, \phi)},$$

we obtain

$$\begin{aligned} \Phi_{m,n}^{as} = & \Phi_{m,n}^{as+} - \Phi_{m,n}^{as-} = \left[e^{2\pi i(m\phi+n\psi)} + e^{2\pi i(m(-\phi+\psi)+n(-3\phi+2\psi))} + e^{2\pi i(m(2\phi-\psi)+n(3\phi-\psi))} + \right. \\ & e^{-2\pi i(m\phi+n\psi)} + e^{-2\pi i(m(-\phi+\psi)+n(-3\phi+2\psi))} + e^{-2\pi i(m(2\phi-\psi)+n(3\phi-\psi))} \left. \right] - \\ & \left[e^{2\pi i(m\phi+n(3\phi-\psi))} + e^{2\pi i(m(-\phi+\psi)+n\psi)} + e^{2\pi i(m(2\phi-\psi)+n(3\phi-2\psi))} + \right. \\ & e^{-2\pi i(m\phi+n(3\phi-\psi))} + e^{-2\pi i(m(-\phi+\psi)+n\psi)} + e^{-2\pi i(m(2\phi-\psi)+n(3\phi-2\psi))} \left. \right], \end{aligned}$$

where the indices are marked by m, n , and angles by ϕ, ψ .

We introduce the following four diagonal matrices

$$\begin{aligned} M_1^+ &= \text{diag}(e^{2\pi i\phi}, e^{2\pi i(-\phi+\psi)}, e^{2\pi i(2\phi-\psi)}, e^{-2\pi i(2\phi-\psi)}, e^{-2\pi i(-\phi+\psi)}, e^{-2\pi i\phi}), \\ M_2^+ &= \text{diag}(e^{2\pi i\psi}, e^{2\pi i(-3\phi+2\psi)}, e^{2\pi i(3\phi-\psi)}, e^{-2\pi i(3\phi-\psi)}, e^{-2\pi i(-3\phi+2\psi)}, e^{-2\pi i\psi}), \\ M_1^- &= \text{diag}(e^{2\pi i(-\phi+\psi)}, e^{2\pi i\phi}, e^{2\pi i(2\phi-\psi)}, e^{-2\pi i(2\phi-\psi)}, e^{-2\pi i(-\phi+\psi)}, e^{-2\pi i\phi}), \\ M_2^- &= \text{diag}(e^{2\pi i\psi}, e^{2\pi i(3\phi-\psi)}, e^{2\pi i(3\phi-2\psi)}, e^{-2\pi i(3\phi-2\psi)}, e^{-2\pi i\psi}, e^{-2\pi i(3\phi-\psi)}). \end{aligned}$$

Then for the function $\Phi_{m,n}^{as}$ we obtain the representation

$$\Phi_{m,n}^{as} = \text{tr}((M_1^+)^m (M_2^+)^n - (M_1^-)^m (M_2^-)^n).$$

The Weyl vector has the form

$$\boldsymbol{\rho} = \frac{1}{2}(\alpha_1 + \alpha_2) = \lambda_1 + \lambda_2.$$

So, according to the definition given above, we introduce new variables

$$x = U_{1,0} = \frac{\Phi_{2,1}^{as}}{\Phi_{1,1}^{as}}, \quad y = U_{0,1} = \frac{\Phi_{1,2}^{as}}{\Phi_{1,1}^{as}}.$$

Taking into account the expression of singular element

$$\begin{aligned} \Phi_{1,1}^{as} = & e^{2\pi i(\phi+\psi)} - e^{2\pi i(-\phi+2\psi)} + e^{2\pi i(-4\phi+3\psi)} - e^{2\pi i(4\phi-\psi)} + e^{2\pi i(5\phi-2\psi)} - e^{2\pi i(5\phi-3\psi)} + \\ & + e^{2\pi i(-5\phi+2\psi)} - e^{2\pi i(-5\phi-3\psi)} + e^{2\pi i(4\phi-3\psi)} - e^{2\pi i(\phi-2\psi)} + e^{2\pi i(-\phi-\psi)} - e^{2\pi i(-4\phi+\psi)}, \end{aligned}$$

after a simple calculation we obtain the following expressions for x, y

$$x = e^{-2\pi i\phi} + e^{2\pi i(\phi-\psi)} + e^{2\pi i(-2\phi+\psi)} + e^{2\pi i(2\phi-\psi)} + e^{2\pi i(-\phi+\psi)} + e^{2\pi i\phi} + 1,$$

$$\begin{aligned} y = & e^{2\pi i(-3\phi+\psi)} + e^{2\pi i(-\psi)} + e^{2\pi i(3\phi-2\psi)} + e^{2\pi i(3\phi-\psi)} + e^{2\pi i(3\phi+2\psi)} + e^{2\pi i\psi} \\ & + e^{2\pi i(-\phi)} + e^{2\pi i(\phi-\psi)} + e^{2\pi i(-2\phi+\psi)} + e^{2\pi i(2\phi-\psi)} + e^{2\pi i(-\phi+\psi)} + e^{2\pi i\phi} + 2. \end{aligned}$$

Turning to the construction of a generating function, we introduce the matrix

$$R_p^\pm = (I_6 - pM_1^\pm)^{-1}, \quad R_q^\pm = (I_6 - qM_2^\pm)^{-1},$$

where I_6 is the identity matrix of order 6, p and q real parameters, and the function

$$\Phi_{p,q}^{as}(\phi, \psi) = \text{tr}(R_p^+ R_q^+ - R_p^- R_q^-).$$

This function factorizes, and the multiplier is the singular element of $\Phi_{1,1}^{as}$.

Computing the ratio $\Phi_{p,q}^{as}/\Phi_{1,1}^{as}$ and representing it in terms of new variables x, y , we come to the generating function for the Chebyshev polynomials of the second kind, associated with the Lie algebra G_2

$$F^{G_2}(p, q; x, y) = \frac{\Phi_{p,q}^{as}}{\Phi_{1,1}^{as}} = (P_1 P_2)^{-1} \left(\sum_{i,j=0}^4 K_{ij} p^i q^j \right).$$

In this expression, the polynomials P_1, P_2 are defined by the formulas

$$P_1 = 1 + (1-x)p + (y+1)p^2 - (x^2 - 2y - 1)p^3 + (y+1)p^4 + (1-x)p^5 + p^6,$$

$$\begin{aligned} P_2 = & 1 + (x-y+1)q + (x^3 - 3xy - 2y - x + 1)q^2 - \\ & (y^2 - 2x^3 + 4xy + 6y + x^2 - 2y + 2x - 1)q^3 + \\ & (x^3 - 3xy - 2y - x + 1)q^4 + (x-y+1)q^5 + q^6, \end{aligned}$$

and non-zero coefficients K_{ij} have the form

$$\begin{aligned}
K_{00} &= 1, K_{10} = 1, K_{01} = x + 1, \\
K_{02} &= x + 1, K_{11} = -x^2 + x + y + 2, \\
K_{03} &= 1, K_{12} = -x^2 + x + 1, K_{21} = y + 1, \\
K_{31} &= 1 - x, K_{13} = 1 - x, K_{22} = -x^2 + xy + y + 2x + 1, \\
K_{41} &= 1, K_{23} = y + 1, K_{32} = -x^2 + x + 1, \\
K_{33} &= -x^2 + x + y + 2, K_{42} = x + 1, \\
K_{34} &= 1, K_{43} = x + 1, K_{44} = 1.
\end{aligned}$$

Using the generating function

$$U_{m,n}(x, y) = \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial^m \partial^n} (F^{G_2}(p, q; x, y)) \Big|_{p=q=0},$$

we write out the first few Chebyshev polynomials of the second kind associated with the root system of the Lie algebra G_2 .

$$\begin{aligned}
U_{0,0} &= 1 \\
U_{1,0} &= x \\
U_{0,1} &= y \\
U_{2,0} &= x^2 - x - y - 1 \\
U_{1,1} &= -x^2 + xy + y + 1 \\
U_{0,2} &= -x^3 + 2xy + y^2 + 2x + y \\
U_{3,0} &= -2xy - x - x^2 - y + x^3 \\
U_{2,1} &= -y + x + x^2 - y^2 - x^3 + x^2y \\
U_{1,2} &= 2x^2 - x - 1 - x^4 + x^2y + x^3 + y^2 + xy^2 \\
U_{0,3} &= -2x^3y + 4xy^2 + 3y^2 + 4xy + 2y - x^3 - x^2 + x^4 - 2x^2y + y^3 \\
U_{4,0} &= y^2 + 2y + x - 3x^2y - x^2 - x^3 + x^4 \\
U_{3,1} &= 2x^3 - 2y^2 - 2x - 2y + x^2y + x^3y - 2xy^2 - 2xy - x^4 + x^2 \\
U_{2,2} &= 1 + 2x^3 - 2y^2 - x - x^5 - y^3 - x^2y + 2x^3y - 2xy^2 - 4xy + x^2y^2 + 2x^4 - 4x^2 \\
U_{1,3} &= -4x^3 + y^2 + 2x + 2x^5 + y^3 + 4x^2y - 4x^3y + 4xy^2 + 6xy + 3x^2y^2 - 2x^4y + xy^3 \\
&\quad - 2x^4 + 2x^2 \\
U_{0,4} &= -1 + 2x^3 + 4y^2 - x - 2y - x^5 + 5y^3 + 6x^2y - 2x^3y + 9xy^2 + 2xy - 2x^4y - 3x^3y^2 \\
&\quad + 6xy^3 - 3x^4 + 3x^2 + x^6 + y^4.
\end{aligned}$$

These polynomials coincide with the polynomials found from recurrence relations in the work [24]

In the presented derivation of generating function is used only the definition of Chebyshev polynomials and the Weyl character formula and absolutely not involved recurrence relations. Since such relations provide several new features, we give in conclusion the following remarks.

The easiest way for obtaining the recurrence relations for Chebyshev polynomials of several variables is to use (8)

$$\Phi_{\mathbf{k}}\Phi_{\mathbf{s}} = \sum_{w \in W} \Phi_{w\mathbf{k}+\mathbf{s}},$$

where the summation in the right hand part applies to all elements of the Weyl group; \mathbf{k} and \mathbf{s} are vectors in the space of the fundamental weights with non-negative integer components $\mathbf{s} = (m, n)$. In the case of two variables, we assume that \mathbf{k} in this space is equal to $\mathbf{k} = (1, 0)$ firstly, and then to $\mathbf{k} = (0, 1)$ and taking in to account the above formula for action generators of the Weyl group on the fundamental weights, we obtain in a convenient normalization

$$\begin{aligned} xU_{m,n} &= U_{m+1,n} + U_{m-1,n+1} + U_{m+2,n-1} + U_{m-2,n+1} + U_{m+1,n-1} + U_{m-1,n} \\ yU_{m,n} &= U_{m,n+1} + U_{m+3,n-1} + U_{m-3,n+2} + U_{m+3,n-2} + U_{m-3,n+1} + U_{m,n-1}. \end{aligned} \quad (18)$$

This relations can be rewritten in the forms

$$\begin{aligned} U_{m,n} &= (x-1)U_{m-1,n} - (y+1)U_{m-2,n} + (x^2-1-2y)U_{m-3,n} - (y+1)U_{m-4,n} \\ &\quad + (x-1)U_{m-5,n} - U_{m-6,n} \\ U_{m,n} &= (-x+y-1)U_{m,n-1} - (x^3-x+1-3xy-2y)U_{m,n-2} \\ &\quad + (-2x^3+x^2+2x-1+4xy+4y+y^2)U_{m,n-3} + (x^3-x+1-3xy-2y)U_{m,n-4} \\ &\quad - (-x+y-1)U_{m,n-5} + U_{m,n-6}, \end{aligned} \quad (19)$$

in each of which changes only a single index. In contrast to the case of Chebyshev polynomials in two variables associated with the Lie algebra A_2 [25], the transition from relations (18) to (19) is not obvious.

As it is shown in [1], this form is closely connected with the polynomials P_1, P_2 , standing in the denominator of the generating function. Namely, the polynomials P_1, P_2 are the characteristic polynomials of matrices M_1^\pm, M_2^\pm . From here the recurrence relation in the form (19) immediately follows.

Let us represent the relations (19) in a matrix form

$$U_{m,n} = U_m M_x^n V_0, \quad U_{m,n} = U_n M_y^n V_0,$$

where U_m and U_n the following row-matrices

$$\begin{aligned} U_m &= (U_{5,n}, -U_{4,n}, U_{3,n}, -U_{2,n}, U_{1,n}, U_{0,n}), \\ U_n &= (U_{m,5}, -U_{m,4}, U_{m,3}, -U_{m,2}, U_{m,1}, U_{m,0}); \end{aligned}$$

V_0 - column matrix

$$V_0 = (0, 0, 0, 0, 0, 1)^T,$$

and matrices M_x, M_y are of the following form

$$M_x = \begin{pmatrix} x-1 & 1 & 0 & 0 & 0 & 0 \\ -(y+1) & 0 & 1 & 0 & 0 & 0 \\ x^2-1-2y & 0 & 0 & 1 & 0 & 0 \\ -(y+1) & 0 & 0 & 0 & 1 & 0 \\ x-1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_y = \begin{pmatrix} -x + y - 1 & 1 & 0 & 0 & 0 & 0 \\ -(x^3 - x + 1 - 3xy - 2y) & 0 & 1 & 0 & 0 & 0 \\ -2x^3 + x^2 + 2x - 1 + 4xy + 4y + y^2 & 0 & 0 & 1 & 0 & 0 \\ -(x^3 - x + 1 - 3xy - 2y) & 0 & 0 & 0 & 1 & 0 \\ -x + y - 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The polynomials P_1, P_2 are the minimal polynomials for the matrices M_x, M_y . Given above relations

$$U_{m,n} = U_m M_x^n V_0, \quad U_{m,n} = U_n M_y^n V_0,$$

can be used for several other variants of receiving of generating functions, and also for some other purposes, for example, to calculate polynomials with negative indices.

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